A simple way of making a Hamiltonian system into a bi-Hamiltonian one

A. Sergyeyev*
The Jacob Blaustein Institute for Desert Research,
Ben-Gurion University of the Negev, Sede Boqer Campus,
84990 Israel

Abstract

Given a Poisson structure (or, equivalently, a Hamiltonian operator) P, we show that its Lie derivative $L_\tau(P)$ along a vector field τ defines another Poisson structure, which is automatically compatible with P, if and only if $[L_\tau^2(P), P] = 0$, where $[\cdot,\cdot]$ is the Schouten bracket. We further prove that if $\dim\ker P \leq 1$ and P is of locally constant rank, then all Poisson structures compatible with a given Poisson structure P on a finite-dimensional manifold M are locally of the form $L_\tau(P)$, where τ is a local vector field such that $L_\tau^2(P) = L_{\tilde\tau}(P)$ for some other local vector field $\tilde\tau$. This leads to a remarkably simple construction of bi-Hamiltonian dynamical systems. We also present a generalization of these results to the infinite-dimensional case. In particular, we provide a new description for pencils of compatible local Hamiltonian operators of Dubrovin–Novikov type and associated bi-Hamiltonian systems of hydrodynamic type.

Key words: compatible Poisson structures, Hamiltonian operators, bi-Hamiltonian systems, integrability, Schouten bracket, master symmetry, Lichnerowicz–Poisson cohomology, hydrodynamic type systems.

MSC 2000: Primary: 37K10; Secondary: 37K05, 37J35.

1 Introduction

The ingenious discovery of Magri [1] (cf. also [2, 3, 4] and the surveys in [5, 6, 7]) that integrable Hamiltonian systems usually prove to be bi-Hamiltonian, and vice versa, leads us to the following fundamental problem: given a (likely to be integrable) dynamical system which is Hamiltonian with respect to a Poisson structure P, how to find another Poisson structure $\tilde P$ compatible with P and such that our system is Hamiltonian with respect to $\tilde P$ as well. This, along with the related problem of classification of compatible Poisson structures, is nowadays a subject of intense research, see e.g. [1]–[26] and references therein.

For the finite-dimensional dynamical systems the results of Lichnerowicz [27] imply that if $\dim\ker P \leq 1$ and P is of locally constant rank (here and below we assume that the vicinities where $\operatorname{rank} P = \text{const}$ are of the same dimension as the underlying manifold), then *all* Poisson structures compatible with P are locally (and, under certain extra conditions, globally) of the form $L_\tau(P)$, i.e., they can be written as Lie derivatives of P along suitable local vector fields τ. Oevel [17] and, independently, Dorfman [5] showed that for invertible P this holds in the infinite-dimensional case as well. Oevel [17, 18] also pointed out that the τ's in question often prove to be *master symmetries* in the sense of [28].

In general $L_\tau(P)$ is not a Poisson structure even if so is P, and it is our goal here to provide a simple description of the 'eligible' τ's, for which $L_\tau(P)$ *is* Poisson. Namely, see Proposition 4 below, we prove that if $\dim\ker P \leq 1$ and P is of locally constant rank, then *all* Poisson structures compatible with P are locally of the form $L_\tau(P)$, where τ is a local vector field such that $L_\tau^2(P) = L_{\tilde\tau}(P)$ holds locally for some other local vector field $\tilde\tau$. For invertible P this holds even in the infinite-dimensional case, and under certain conditions τ and $\tilde\tau$ are global, see Proposition 3 and Section 6 below for details. This improves earlier results of Petalidou, who found a criterion for $L_\tau(P)$ to be a Poisson structure under the assumption that P is nondegenerate, see Proposition 3.1 of [26], and also [25]. Moreover, Propositions 3 and 4 yield a criterion

*On leave from Silesian University in Opava, Mathematical Institute, Na Rybníčku 1, 746 01 Opava, Czech Republic. E-mail: Artur.Sergyeyev@math.slu.cz

for existence of a regular or weakly regular Poisson pencil such that a given dynamical system is locally bi-Hamiltonian with respect to this pencil, see Remark 1 and Corollary 5.

On the other hand, if both P and $L_\tau(P)$ *are* Poisson structures, then they are *automatically* compatible [27, 17, 20, 5], no matter whether P is (non)degenerate, so it is natural to ask when $L_\tau(P)$ is a Poisson structure if so is P. We show that this is the case if and only if $[L_\tau^2(P), P] = 0$, where $[\cdot,\cdot]$ is the Schouten bracket, see Proposition 1 below for details. Note that this result first appeared in the earlier version [29] of the present paper and was recently rediscovered by Chavchanidze [30] in the context of non-Noether symmetries.

By Corollary 1 if there exist a (local) vector field $\tilde\tau$ and a constant α such that $L_\tau^2(P) = L_{\tilde\tau}(P) + \alpha P$, then $[L_\tau^2(P), P] = 0$, and thus $L_\tau(P)$ is a Poisson structure. For $\alpha = 0$ and $\tilde\tau = 0$ we recover the result of Smirnov [24], cf. also formulae (2.8) in Magri [21] and the papers [22, 23]; in this case P and $L_\tau(P)$ form the so-called *exact bi-Hamiltonian structure*, and τ is called a *Liouville vector field* [22, 23]. On the other hand, given two compatible Poisson structures P and $\tilde P$ that share a common scaling and assuming that one of them, P, is nondegenerate, we can readily construct τ and $\tilde\tau$ such that the second bivector of the pair, $\tilde P$, can be written as $L_\tau(P)$ and $L_\tau^2(P) = L_{\tilde\tau}(P)$, see Proposition 2 and Corollary 3 below.

As an application of our results, in Section 6 we give a new description of compatible Poisson structures of Dubrovin–Novikov [31, 32] type and of associated bi-Hamiltonian systems of hydrodynamic type.

2 Basic definitions

Let M be a smooth finite-dimensional manifold. Below we assume all objects to be smooth enough for all the required derivatives to exist.

Recall that a *bivector* on M is a skew-symmetric contravariant tensor field of rank two. To any bivector B there corresponds, in a natural way, a skew-symmetric linear operator (which for the sake of simplicity will be denoted by the same letter) $B : \wedge^1 M \to TM$. A bivector B is called *nondegenerate*, and the associated operator B is called *invertible*, if $\ker B = \{\chi \in \wedge^1 M : B\chi = 0\}$ is exhausted by $\chi = 0$.

The Schouten bracket $[H, K]$ of two bivectors H and K is given by the formula (see e.g. Section 2.8 of [5])

$$[H, K](\xi_1, \xi_2, \xi_3) = \langle H L_{K\xi_1}(\xi_2), \xi_3\rangle + \langle K L_{H\xi_1}(\xi_2), \xi_3\rangle + \text{cycle}(1, 2, 3),$$

where $\xi_1, \xi_2, \xi_3 \in \wedge^1 M$, L_X stands for the Lie derivative along a vector field X, and $\langle\cdot,\cdot\rangle$ denotes the natural pairing of vector fields and one-forms on M.

Thus, $[H, K]$ is an antisymmetric contravariant tensor of rank three, i.e., a trivector, and its components in local coordinates read [5]

$$[H, K]^{ijk} = -\sum_{m=1}^{\dim M} \left(K^{mk} \frac{\partial H^{ij}}{\partial x^m} + H^{mk} \frac{\partial K^{ij}}{\partial x^m} + \text{cycle}(i, j, k) \right).$$

It is well known [20, 5] that

$$[H, K] = [K, H], \tag{1}$$

and for any vector field τ on M

$$L_\tau([H, K]) = [L_\tau(H), K] + [H, L_\tau(K)]. \tag{2}$$

Recall that in local coordinates the Lie derivative of a bivector P along a vector field τ reads

$$(L_\tau(P))^{ij} = \sum_{k=1}^{\dim M} \left(\tau^k \frac{\partial P^{ij}}{\partial x^k} - P^{kj} \frac{\partial \tau^i}{\partial x^k} - P^{ik} \frac{\partial \tau^j}{\partial x^k} \right).$$

If $[P, P] = 0$, then a bivector P on M is called a *Poisson bivector* or, if it is perceived as an operator $P : \wedge^1 M \to TM$, a *Hamiltonian* [5] or *implectic* [4] operator. The associated Poisson bracket reads $\{f, g\}_P = \langle df, Pdg\rangle$, where f and g are smooth functions on M, see e.g. [5, 7]. A pair (M, P), where P is a Poisson bivector on M, is called a *Poisson manifold* [27].

Two Poisson bivectors P_0 and P_1 (or the associated Hamiltonian operators) are said [1, 2, 3] to be *compatible* (or to form a *Hamiltonian pair*), if any linear combination of P_0 and P_1 is again a Poisson bivector. It is well known, see e.g. [1, 2, 3, 5], that P_0 and P_1 are compatible if and only if $[P_0, P_1] = 0$.

3 When $L_\tau(P)$ is a Poisson bivector?

Proposition 1 *Let P be a Poisson bivector and τ be a vector field on M. Then $L_\tau(P)$ is a Poisson bivector, which is automatically compatible with P, if and only if*

$$[L_\tau^2(P), P] = 0. \tag{3}$$

Proof. Writing out the identity $L_\tau([P,P]) = 0$ with usage of (1) and (2) yields (cf. Proposition 7.8 from [5])

$$[P,P] = 0 \Rightarrow [L_\tau(P), P] = 0. \tag{4}$$

Next, using (2) and (4), we can rewrite the identity $L_\tau^2([P,P]) = 0$ as $[L_\tau^2(P), P] + [L_\tau(P), L_\tau(P)] = 0$. As $L_\tau(P)$ is a Poisson bivector if and only if $[L_\tau(P), L_\tau(P)] = 0$, the result immediately follows. □

For instance, let $M = \mathbb{R}^{2m+1}$ with (global) coordinates x^i and h be a smooth function on M. Taking for P the canonical Poisson structure of maximal rank on M and setting $\tilde{P} = L_\tau(P)$ for $\tau = (-(x^1)^2/2, \dots, -(x^m)^2/2, 0, \dots, 0, -h/2)^T$, where the superscript T stands for the transposed matrix, we have

$$P = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{P} = \begin{pmatrix} 0 & \Lambda & X^T \\ -\Lambda & 0 & -Y^T \\ -X & Y & 0 \end{pmatrix}.$$

Here I is the $m \times m$ unit matrix, $\Lambda = \operatorname{diag}(x^1, \dots, x^m)$ is an $m \times m$ diagonal matrix, $X = (\partial h/\partial x^{m+1}, \dots, \partial h/\partial x^{2m})$ and $Y = (\partial h/\partial x^1, \dots, \partial h/\partial x^m)$ are m-component columns.

For $h = \sum_{i=1}^m f_i(x^i, x^{i+m})/\Delta_i - x^{2m+1}\sum_{i=1}^m x^i$, where f_i are arbitrary smooth functions of their arguments, and $\Delta_i = \prod_{j=1, j\neq i}^m (x^i - x^j)$, this construction yields a pair of Poisson structures arising in the theory of the so-called one-Casimir chains [7, 33, 34]. It is straightforward to verify that in this case $\tilde{P}$ indeed is a Poisson structure by virtue of (3), and P and $\tilde{P}$ are compatible by (4). The examples of integrable systems associated with this pair can be found in [7, 33, 34].

Remarkably, (3) is often easier to verify than $[L_\tau(P), L_\tau(P)] = 0$. In particular, using (4) and bilinearity of the Schouten bracket readily yields the following result.

Corollary 1 *Let P be a Poisson bivector and there exist vector fields τ and $\tilde{\tau}$ and a constant α such that*

$$L_\tau^2(P) = L_{\tilde{\tau}}(P) + \alpha P. \tag{5}$$

Then $L_\tau(P)$ is a Poisson bivector, which is automatically compatible with P.

If we set $\alpha = \tilde{\tau} = 0$ in (5), then we recover Proposition 4.1 of Smirnov [24] (cf. also Section 2 of [23]). Note that for P being a Poisson structure of Dubrovin–Novikov type [31, 32] the condition (5) with $\alpha = \tilde{\tau} = 0$ was also studied by Fordy and Mokhov [12].

For an example of somewhat different kind, let $M = \mathbb{R}^3$ with the (global) coordinates x, y, z and $\tau = (-x - x^3/3, 0, zx^2 - x^2y^2)^T$. Take for P the canonical Poisson structure on $\mathbb{R}^3$. Then we have

$$P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, L_\tau(P) = \begin{pmatrix} 0 & 1 + x^2 & 2x^2y \\ -1 - x^2 & 0 & 2xz - 2xy^2 \\ -2x^2y & -2xz + 2xy^2 & 0 \end{pmatrix}. \tag{6}$$

In turn, $L_\tau^2(P) = L_{\tilde{\tau}}(P)$, where $\tilde{\tau} = (-x^5/15 - x, 0, (zx^3 - x^4y^2)/3)^T$, so $L_\tau(P)$ is a Poisson bivector by Corollary 1. This pair of compatible Poisson bivectors was found in [35] along with a related completely integrable bi-Hamiltonian system.

Now assume that we have two vector fields τ_1 and τ_2 meeting the requirements of Proposition 1. When are the Poisson bivectors $P_1 = L_{\tau_1}(P)$ and $P_2 = L_{\tau_2}(P)$ compatible?

Corollary 2 *Given a Poisson bivector P and two vector fields τ_1, τ_2 meeting the requirements of Proposition 1, the Poisson bivectors $L_{\tau_1}(P)$ and $L_{\tau_2}(P)$ are compatible if and only if $[L_{\tau_1}(L_{\tau_2}(P)), P] = 0$.*

Proof. We have the following identity:

$$0 = L_{\tau_i}([L_{\tau_{3-i}}(P), P]) = [L_{\tau_i}(L_{\tau_{3-i}}(P)), P] + [L_{\tau_i}(P), L_{\tau_{3-i}}(P)], \quad i = 1, 2. \tag{7}$$

Equations (7) for $i = 1$ and $i = 2$ are equivalent. Indeed, subtracting (7) with $i = 1$ from (7) with $i = 2$, and using the symmetry property (1) of the Schouten bracket and the equality $L_{\tau_1}L_{\tau_2} - L_{\tau_1}L_{\tau_2} = L_{[\tau_1,\tau_2]}$, where $[\cdot,\cdot]$ stands for the commutator of vector fields, we obtain $[L_{[\tau_1,\tau_2]}(P), P] = 0$. But this holds automatically by virtue of (4). Thus, $[L_{\tau_1}(P), L_{\tau_2}(P)] = 0$ if and only if $[L_{\tau_1}(L_{\tau_2}(P)), P] = 0$. □

If $[\tau_1, \tau_2] = 0$, $L_{\tau_1}(L_{\tau_2}(P)) = 0$, and $L^2_{\tau_i}(P) = 0$, $i = 1, 2$, we recover Proposition 4.2 of Smirnov [24].

For nondegenerate P it is always possible [17] to find locally (and even globally, if the symplectic form associated with P^{-1} is exact) a 'scaling' vector field Z such that $L_Z(P) = P$. Then we can replace $\tilde\tau$ by $\tilde\tau + \alpha Z$ and assume without loss of generality that $\alpha = 0$ in (5). As we shall see below, if $\dim\ker P \le 1$ and P is of locally constant rank, then the construction of Corollary 1 with $\alpha = 0$ locally (and globally too, if the second de Rham cohomology of M is trivial and P is nondegenerate) yields *all* Poisson bivectors compatible with P.

Note that (5) with $\alpha = 0$ often holds for the Poisson bivectors associated with integrable systems, provided τ and $\tilde\tau$ are 'first' and 'second' master symmetries for the latter, cf. e.g. [5, 7, 17, 18, 28]. Moreover, in presence of a scaling symmetry we can easily construct the corresponding τ and $\tilde\tau$ using the results of Oevel [17, 18]. Namely, the following assertions hold.

Proposition 2 *Assume that P and $\tilde P$ are Poisson bivectors, P is nondegenerate, and there exists a 'scaling' vector field τ_0 such that $L_{\tau_0}(P) = \mu P$ and $L_{\tau_0}(\tilde P) = \nu \tilde P$ for some constants μ and ν, $\mu \ne \nu/2$. Then P and $\tilde P$ are compatible if and only if*

$$L_{R\tau_0}(P) = (2\mu - \nu)\tilde P, \tag{8}$$

where $R = \tilde P P^{-1}$.

Proof. If $[P, \tilde P] = 0$, then (8) holds by Theorem 2 of Oevel [18]. On the other hand, if (8) holds, then P and $\tilde P$ are compatible by (4) with $\tau = (1/(2\mu - \nu))R\tau_0$, as $\tilde P = L_\tau(P)$. □

Corollary 3 *Let P be a nondegenerate Poisson bivector and $\tilde P$ be a bivector on M, and let there exist a 'scaling' vector field τ_0 such that $L_{\tau_0}(P) = \mu P$, $L_{\tau_0}(\tilde P) = \nu\tilde P$, and (8) holds with $R = \tilde P P^{-1}$ for some constants μ and ν such that $\mu \ne \nu/2$ and $\mu \ne 2\nu/3$. Then $\tilde P$ is a Poisson bivector, which is automatically compatible with P, if and only if*

$$L^2_{R\tau_0}(P) = \frac{2\mu - \nu}{3\mu - 2\nu} L_{R^2\tau_0}(P). \tag{9}$$

Proof. If $\tilde P$ is a Poisson bivector, then it is compatible with P by virtue of (8) and (4), and (9) holds by Theorem 2 of Oevel [18]. On the other hand, let (9) hold. By virtue of (8) we have $\tilde P = L_\tau(P)$ for $\tau = R\tau_0/(2\mu - \nu)$, so (9) implies that (5) holds for $\tilde\tau = R^2\tau_0/((2\mu-\nu)(3\mu-2\nu))$, and $\tilde P$ is Poisson by Corollary 1. □

Let e.g. $M = \mathbb{R}^{2m}$ with (global) coordinates $x^1, \dots, x^{2m}$, and

$$P_r = \begin{pmatrix} 0 & \Lambda_r \\ -\Lambda_r & 0 \end{pmatrix},$$

where $\Lambda_r = \mathrm{diag}((x^1)^r, \dots, (x^m)^r)$ is a diagonal $m \times m$ matrix. Set $\tau_0 = (x^1, \dots, x^m, 0, \dots, 0)^T$. We have $L_{\tau_0}(P_r) = (r-1)P_r$. As P_0 obviously is a Poisson bivector and (8) and (9) hold for $P = P_0$, $\tilde P = P_r$, $\mu = -1$ and $\nu = r - 1$, by Corollary 3 for any r P_r is a Poisson bivector compatible with P_0, and $P_r = L_{\tau_r}(P_0)$, where $\tau_r = (-(x^1)^{r+1}/(r+1), \dots, -(x^m)^{r+1}/(r+1), 0, \dots, 0)^T$. Clearly, $[L_{\tau_r}(L_{\tau_s}(P_0)), P_0] = 0$, so the Poisson bivectors P_r and P_s are compatible for any r and s by Corollary 2.

4 Compatibility and Lichnerowicz–Poisson cohomology

The condition $[L^2_\tau(P), P] = 0$ is intimately related to the so-called Lichnerowicz–Poisson cohomology introduced in [27]. Indeed, the second Lichnerowicz–Poisson cohomology $H^2_P(M)$ of a Poisson manifold (M, P) is precisely the set of bivectors B solving $[B, P] = 0$ modulo the solutions of the form $B = L_Y(P)$, where Y

is a vector field on M. Hence, $[L_\tau^2(P), P] = 0$ if and only if there exist a vector field $\tilde{\tau}$ on M and constants a_i such that

$$L_\tau^2(P) = L_{\tilde{\tau}}(P) + \sum_{i=1}^{d} a_i B_i, \tag{10}$$

where B_i, $i = 1, \ldots, d \equiv \dim H_P^2(M)$, form a basis of $H_P^2(M)$. Thus, if $H_P^2(M)$ is known (see e.g. [36] for a survey of results on its computation), Eq. (10) yields a simple way to verify whether $L_\tau(P)$ is Poisson.

For instance, if P is nondegenerate, $H_P^2(M)$ is isomorphic [27] to the second de Rham cohomology $H^2(M)$ of M, so the following assertion holds.

Proposition 3 *Suppose that $H^2(M) = 0$, and let P be a nondegenerate Poisson bivector on M. Then a bivector $\tilde{P}$ on M is a Poisson bivector compatible with P if and only if there exist vector fields τ and $\tilde{\tau}$ on M such that $\tilde{P} = L_\tau(P)$ and $L_\tau^2(P) = L_{\tilde{\tau}}(P)$.*

Proof. For nondegenerate P the condition $[B, P] = 0$ is equivalent, cf. e.g. [19, 27], to $d(P^{-1}BP^{-1}) = 0$. As $H^2(M) = 0$, this is equivalent to $P^{-1}BP^{-1} = d\gamma_B$ for some one-form $\gamma_B \in \wedge^1 M$. Upon setting $Y_B = -P\gamma_B$ we have $B = L_{Y_B}(P)$, see e.g. [17, 19]. Setting $\tau = Y_{\tilde{P}}$ and $\tilde{\tau} = Y_K$, where $K = L_\tau^2(P)$, and using Proposition 1 completes the proof. □

As the second de Rham cohomology is always locally trivial, Proposition 3 *locally* describes *all* Poisson bivectors compatible with a nondegenerate P even if $H^2(M) \neq 0$. Moreover, if $\dim\ker P \leq 1$ on the whole of M and P is of locally constant rank, then $H_P^2(M)$ is locally trivial [27], i.e., for any bivector B satisfying $[B, P] = 0$ there still exists a local (but not necessarily global) vector field Y_B such that $B = L_{Y_B}(P)$, so we arrive at the following result.

Proposition 4 *Let P be a Poisson bivector of locally constant rank such that* $\dim\ker P \leq 1$ *everywhere on M. Then a bivector $\tilde{P}$ on M is a Poisson bivector compatible with P if and only if there exist local vector fields τ and $\tilde{\tau}$ such that the equalities $L_\tau^2(P) = L_{\tilde{\tau}}(P)$ and $\tilde{P} = L_\tau(P)$ hold locally.*

By passing from the system of equations $[L_\tau^2(P), P] = 0$ for τ to $L_\tau^2(P) = L_{\tilde{\tau}}(P)$ we have essentially partially integrated the former, as $L_\tau^2(P) = L_{\tilde{\tau}}(P)$ is a second order system of differential equations with respect to τ, while $[L_\tau^2(P), P] = 0$ is of third order. Thus, solving $L_\tau^2(P) = L_{\tilde{\tau}}(P)$ instead of $[L_\tau^2(P), P] = 0$ can considerably simplify the search for and the classification of Poisson bivectors compatible with P.

5 Construction of integrable bi-Hamiltonian dynamical systems

A vector field X on M is called *locally Hamiltonian* with respect to a Poisson bivector P, if $L_X(P) = 0$, and *globally Hamiltonian* w.r.t. P, if it is locally Hamiltonian w.r.t. P and there exists a (smooth) function H on M such that $X = PdH$, cf. e.g. [5, 7, 24] and references therein.

Likewise, a vector field X on M is called *locally bi-Hamiltonian* with respect to a pair of compatible Poisson bivectors P and $\tilde{P}$, if $L_X(P) = L_X(\tilde{P}) = 0$ on M, cf. e.g. [1, 3, 7]. Finally, X is *globally bi-Hamiltonian*, if it is locally bi-Hamiltonian and there exist smooth functions H and $\tilde{H}$ on M such that $X = PdH = \tilde{P}d\tilde{H}$.

Corollary 4 *Consider a Poisson bivector P and a vector field X on M such that $L_X(P) = 0$. Assume that there exists a vector field τ on M such that $L_\tau(P) \neq 0$, $L_X(L_\tau(P)) = 0$, and $[L_\tau^2(P), P] = 0$. Then X is locally bi-Hamiltonian with respect to P and $\tilde{P} \equiv L_\tau(P)$. If there also exist globally defined smooth functions H and $\tilde{H}$ on M such that $X = PdH = \tilde{P}d\tilde{H}$, then X is globally bi-Hamiltonian on M.*

This result generalizes Theorem 5.1 of Smirnov [24] and Corollary 1 of Chavchanidze [30].

For instance, if we take P from (6) and $\tau = (-x - x^3/3, 0, zx^2 - x^2y^2)^T$, as in Section 3, then $X = PdH$ with $H = x^2y^2 - zx^2 - z$ meets the requirements of Corollary 4 and is globally bi-Hamiltonian: $X = L_\tau(P)d\tilde{H}$ with $\tilde{H} = z$, cf. [7, 35].

We can invoke Corollaries 1 or 3 or Propositions 3 or 4 in order to verify the condition $[L_\tau^2(P), P] = 0$. If $H^2(M) = 0$ and P is nondegenerate, then by virtue of Proposition 3 the conditions of Corollary 4 are not just sufficient but also *necessary* for a Hamiltonian w.r.t. P vector field X to be bi-Hamiltonian, assuming that

P is one of two compatible Poisson bivectors. Moreover, in this case Proposition 3 enables us to replace the condition $[L^2_\tau(P), P] = 0$ by the requirement of existence of a vector field $\tilde{\tau}$ on M such that $L^2_\tau(P) = L_{\tilde{\tau}}(P)$.

Likewise, Proposition 4 yields the following result.

Corollary 5 *Consider a Poisson bivector P of locally constant rank and a vector field X on M such that $L_X(P) = 0$ and $\dim\ker P \leq 1$ everywhere on M. Then X is locally bi-Hamiltonian with respect to P and some other Poisson bivector $\tilde{P}$ if and only if there exist local vector fields τ and $\tilde{\tau}$ such that locally we have $L_\tau(P) \neq 0$, $L_X(L_\tau(P)) = 0$, $L^2_\tau(P) = L_{\tilde{\tau}}(P)$, and $\tilde{P}$ can be locally written as $L_\tau(P)$.*

Under the assumptions of Corollary 4 or 5, suppose that at least one of the Poisson bivectors P and $\tilde{P}$ is nondegenerate, so the dimension of M is even. Denote the nondegenerate bivector by P_0, and let P_1 stand for the remaining Poisson bivector. Then [3, 4] $R = P_1 P_0^{-1}$ is a hereditary recursion operator for the dynamical system $\dot{x} = X$, and the eigenvalues of R provide involutive integrals for $\dot{x} = X$ [3]. In particular, if R has a maximal possible number (i.e., $(1/2)\dim M$) of distinct eigenvalues, and all of them are functionally independent, then these eigenvalues form [3] a complete set of functionally independent involutive integrals for the dynamical system $\dot{x} = X$, ensuring its complete integrability in the sense of Liouville's theorem.

Note that Corollaries 4 and 5 readily generalize to the case of *quasi-bi-Hamiltonian systems* considered e.g. in [7, 37], when we have $L_{\rho X}(\tilde{P}) = 0$ for some smooth function ρ on M instead of $L_X(\tilde{P}) = 0$: it suffices to replace the conditions $L_X(L_\tau(P)) = 0$ and $X = \tilde{P}d\tilde{H}$ by $L_{\rho X}(L_\tau(P)) = 0$ and $X = (1/\rho)\tilde{P}d\tilde{H}$ respectively.

Remark 1 Recall that given two compatible Poisson bivectors P_0 and P_1, a Poisson pencil $\mathcal{P}$ associated with P_0 and P_1 is the set of all linear combinations of the form $\lambda P_0 + \mu P_1$, where λ and μ are constants. The compatibility of P_0 and P_1 implies that any such linear combination again is a Poisson bivector. A vector field X is said to be locally bi-Hamiltonian with respect to the Poisson pencil associated with P_0 and P_1 if $L_X(\lambda P_0 + \mu P_1) = 0$ for any λ and μ.

Let us call a Poisson pencil $\mathcal{P}$ *regular* (resp. *weakly regular*) if there exist λ and μ such that $P = \lambda P_0 + \mu P_1$ is nondegenerate (resp. P is of locally constant rank on M and $\dim\ker P \leq 1$ everywhere on M). Let $\tilde{P} = \tilde{\lambda} P_0 + \tilde{\mu} P_1$ be linearly independent of P. Then Proposition 3 yields the following result:

Suppose that $H^2(M) = 0$. Then a vector field X on M is locally bi-Hamiltonian with respect to a regular Poisson pencil $\mathcal{P}$ if and only if $L_X(P) = 0$ and there exist vector fields τ and $\tilde{\tau}$ such that $L_\tau(P) \neq 0$, $L_X(L_\tau(P)) = 0$, $L^2_\tau(P) = L_{\tilde{\tau}}(P)$, and $\tilde{P}$ can be written as $L_\tau(P)$.

Likewise, Corollary 5 is equivalent to the following assertion:

A vector field X on M is locally bi-Hamiltonian with respect to a weakly regular Poisson pencil $\mathcal{P}$ if and only if $L_X(P) = 0$ and there exist local vector fields τ and $\tilde{\tau}$ such that locally we have $L_\tau(P) \neq 0$, $L_X(L_\tau(P)) = 0$, $L^2_\tau(P) = L_{\tilde{\tau}}(P)$, and $\tilde{P}$ can be locally written as $L_\tau(P)$.

Thus, we have obtained necessary and sufficient conditions for existence of a (weakly) regular Poisson pencil such that a given vector field X is locally bi-Hamiltonian with respect to this pencil.

6 Infinite-dimensional case

Up to now we assumed that M is a finite-dimensional manifold. Nevertheless, all the above results, except for Proposition 4, Corollary 5 and the statement on Liouville integrability of $\dot{x} = X$, readily extend to the case of Hamiltonian formalism over the complex over a Lie algebra $\mathfrak{A}$ associated with a representation π of $\mathfrak{A}$, see Example 2.2 of [5]. This setting is very general and naturally includes the most of interesting cases when the underlying manifold M is infinite-dimensional.

The desired extension is achieved by passing from bivectors to the skew-symmetric operators $P : \Omega^1 \to \mathfrak{A}$, where Ω^1 is the set of all linear mappings from $\mathfrak{A}$ to π, and replacing a) the notion of nondegeneracy of a bivector by that of invertibility of the operator, b) the condition $H^2(M) = 0$ by the requirement of triviality of the second cohomology for the complex in question. The standard Hamiltonian formalism over a finite-dimensional manifold M is recovered if we take for the complex in question the de Rham complex of M [5].

Moreover, Propositions 1 and 2 and Corollaries 1–4 in fact remain valid (after performing the above replacement) within the framework of Hamiltonian formalism over an arbitrary (Ω, d)-complex over a Lie algebra $\mathfrak{A}$ with nondegenerate pairing between $\mathfrak{A}$ and Ω^1, see Ch. 2 of [5] for more details on such complexes.

The key example of an infinite-dimensional (Ω, d)-complex undoubtedly is that of formal calculus of variations, see e.g. [5] for further details, and cf. [38] for a somewhat different approach to the Hamiltonian formalism for PDEs. Let us briefly recall some basic properties of this complex for the case of one space variable x ($x \in \mathbb{R}$ or $x \in S^1$) and n dependent variables, essentially following [5, 6].

Consider an algebra $\mathcal{A}_j$ of locally analytic functions of $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_j$, where $\boldsymbol{u}_k = (u_k^1, \ldots, u_k^n)^T$ are n-component vectors, $\boldsymbol{u}_0 \equiv \boldsymbol{u}$, and let $\mathcal{A} = \bigcup_{j=0}^{\infty} \mathcal{A}_j$. We shall call the elements of $\mathcal{A}$ *local* functions. Let us make $\mathcal{A}$ into a differential algebra by introducing a derivation

$$D \equiv D_x = \partial/\partial x + \sum_{j=0}^{\infty} \boldsymbol{u}_{j+1} \partial/\partial \boldsymbol{u}_j.$$

Denote by $\operatorname{Im} D$ the image of D in $\mathcal{A}$, and let $\tilde{\mathcal{A}} = \mathcal{A}/\operatorname{Im} D$. Following the tradition, denote the canonical projection $\rho : \mathcal{A} \to \tilde{\mathcal{A}}$ by $\int dx$. Then we have the following 'formal integration by parts' formula valid for any $f, g \in \mathcal{A}$:

$$\int f D(g) dx = -\int g D(f) dx.$$

Informally, x plays the role of the space variable, and D is the total x-derivative, cf. e.g. [6], so one often writes $\partial^j \boldsymbol{u}/\partial x^j$ instead of $\boldsymbol{u}_j$.

We take for $\mathfrak{A}$ the Lie algebra of evolution vector fields (EVFs) of the form $X = \sum_{m=0}^{\infty} \sum_{p=1}^{n} D^m(h^p) \partial/\partial u_m^p$; $\boldsymbol{h} = (h^1, \ldots, h^n)^T$ is called the characteristics of X, $h^p \in \mathcal{A}$. The characteristics of the commutator $[X, Y]$ of two EVFs is given by $Y(\boldsymbol{h}) - X(\boldsymbol{g})$, where $\boldsymbol{h}$ and $\boldsymbol{g}$ are characteristics of X and Y, respectively, see e.g. [6]. Clearly, the characteristics are in one-to-one correspondence with the EVFs, so in what follows we shall identify the EVFs with their characteristics, cf. e.g. [6].

We have $\Omega^0 = \tilde{\mathcal{A}}$, and Ω^1 consists of the vertical one-forms $\omega = \sum_{p=1}^{n} \gamma_p du^p$, where $\gamma_p \in \mathcal{A}$. The pairing between $\mathfrak{A}$ and Ω^1 is given by

$$\langle X, \omega \rangle = \int \sum_{p=1}^{n} \gamma_p h^p dx.$$

Introduce in $\mathcal{A}$ the operator of variational derivative (see e.g. [5, 6])

$$\delta/\delta \boldsymbol{u} = \sum_{m=0}^{\infty} (-D)^m \partial/\partial \boldsymbol{u}_m.$$

Then the differential of $F = \int f dx \in \tilde{\mathcal{A}}$ reads $dF = \sum_{r=1}^{n} \delta f/\delta u^r du^r$.

Let P be a Hamiltonian operator of the form

$$P = \sum_{m=0}^{q} a_m D^m + \sum_{\kappa=1}^{p} \boldsymbol{G}_\kappa \otimes D^{-1} \circ \boldsymbol{\gamma}_\kappa,$$

where $q \geq 0$, a_i are $s \times s$ matrices with entries from $\mathcal{A}$, $\boldsymbol{G}_\kappa, \boldsymbol{\gamma}_\kappa \in \mathcal{A}^s$ (in fact, $\boldsymbol{G}_\kappa, \boldsymbol{\gamma}_\kappa \in \mathfrak{A}$). Then, or even more broadly, for P that can be written as a formal series of the form $\sum_{m=-\infty}^{q} a_m D^m$, Propositions 1, 2 and 4 and Corollaries 1–5 remain valid, if we replace the requirement of nondegeneracy of P or the condition of local constancy of rank of P along with $\dim \ker P \leq 1$ by $\det a_q \neq 0$ and allow for nonlocalities (like e.g. $D^{-1}(\zeta)$ for some $\zeta \in \mathcal{A}$) in the characteristics of τ and $\tilde{\tau}$.

Consider e.g. the operators of the form

$$P^{ij} = g^{ij}(\boldsymbol{u}) D + \sum_{k=1}^{n} b_k^{ij}(\boldsymbol{u}) u_x^k + \sum_{\alpha=1}^{r} \sum_{k,l=1}^{n} \epsilon_\alpha (w^\alpha(\boldsymbol{u}))_k^i u_x^k D^{-1} \circ (w^\alpha(\boldsymbol{u}))_l^j u_x^l,$$

$$\tilde{P}^{ij} = \tilde{g}^{ij}(\boldsymbol{u}) D + \sum_{k=1}^{n} \tilde{b}_k^{ij}(\boldsymbol{u}) u_x^k + \sum_{\alpha=1}^{\tilde{r}} \sum_{k,l=1}^{n} \tilde{\epsilon}_\alpha (\tilde{w}^\alpha(\boldsymbol{u}))_k^i u_x^k D^{-1} \circ (\tilde{w}^\alpha(\boldsymbol{u}))_l^j u_x^l,$$

where ϵ_α and $\tilde{\epsilon}_\alpha$ are constants satisfying $(\epsilon_\alpha)^2 = 1$ and $(\tilde{\epsilon}_\alpha)^2 = 1$.

These operators have a common scaling $\tau_0 = x\boldsymbol{u}_x$. Hence, if $\det g^{ij} \neq 0$, Proposition 2 and Corollary 3 with $\tau_0 = x\boldsymbol{u}_x$ and $\mu = \nu = 1$ provide easily verifiable criteria for compatibility of P and $\tilde{P}$ and for $\tilde{P} = L_\tau(P)$ to be a Hamiltonian operator if so is P. The operators of this type were introduced by Ferapontov [39], and we refer the reader to this paper for the conditions under which these operators are Hamiltonian and the discussion of their properties.

Consider now the complex of formal calculus of variations for the case of two dependent variables $u^1 \equiv u$ and $u^2 \equiv v$ (so $u_x \equiv u^1_1$, $v_x \equiv u^2_1$), and two skew-symmetric operators [40]

$$P = \begin{pmatrix} 0 & D \\ D & 2D \end{pmatrix}, \tilde{P} = \begin{pmatrix} 2uD + u_x & -D^2 + vD + 2uD + 2u_x \\ D^2 + vD + v_x + 2uD & 4vD + 2v_x \end{pmatrix}.$$

Informally, the role of manifold M is played here by an appropriate functional space (e.g. the Schwartz space) of two-component smooth functions (u, v) of x, cf. e.g. [5, 7].

For $\tau_0 = (xu_x + u, xv_x + v)^T$ we have $L_{\tau_0}(P) = 0$ and $L_{\tau_0}(\tilde{P}) = \tilde{P}$. The operator P is obviously Hamiltonian, as it is of odd order and has constant coefficients, cf. e.g. [5, 6]. Since (8) and (9) are easily seen to hold with $\mu = 0$ and $\nu = 1$, our Corollary 3 reconfirms that $\tilde{P}$ is [40] a Hamiltonian operator compatible with P.

What is more [40], $X = (\frac{1}{2}D(-u_x + 2uv - u^2), \frac{1}{2}D(v_x - 2u_x - 2u^2 + 2uv + v^2))^T$ is Hamiltonian with respect to P: $X = PdH$ for $H = (1/2)\int(-u_x v - u^2 v + uv^2)dx$. We also have $\tilde{P} = L_\tau(P)$, where $\tau = -\tilde{P}\gamma$, $\gamma = (-2xu + xv, xu)^T$, and $L_X(\tilde{P}) = 0$, so by Corollary 4 X is locally bi-Hamiltonian with respect to P and $\tilde{P} = L_\tau(P)$. In fact, X is globally bi-Hamiltonian [40], as $X = \tilde{P}d\tilde{H}$ for $\tilde{H} = (1/2)\int(uv - v^2)dx$.

Thus, we have reconfirmed (cf. [40]) the bi-Hamiltonian nature of the modified dispersive water wave system

$$u_t = \frac{1}{2}D(-u_x + 2uv - u^2), \quad v_t = \frac{1}{2}D(v_x - 2u_x - 2u^2 + 2uv + v^2).$$

For another example, consider the complex of formal calculus of variations [5] for the case of one space variable x and n dependent variables u^i, and a skew-symmetric operator P of Dubrovin–Novikov type [31, 32], cf. also [39],

$$P^{ij} = g^{ij}(\boldsymbol{u})D + \sum_{k=1}^{n} b^{ij}_k(\boldsymbol{u})u^k_x, \tag{11}$$

where $\boldsymbol{u} = (u^1, \ldots, u^n)^T$, $u^k_x \equiv u^k_1$, and the indices $i, j, k, \ldots$ run from 1 to n. The role of the manifold M is now played by the loop space, i.e., the space of smooth mappings from S^1 with a local coordinate x to an n-dimensional manifold N with local coordinates u^i, see e.g. [13, 31, 32] for details.

Recall [31, 32] that P (11) with $\det g^{ij} \neq 0$ is a Hamiltonian operator if and only if g^{ij} is a flat (pseudo-) Riemannian metric on N and $b^{ij}_k = -\sum_{m=1}^n g^{im}\Gamma^j_{mk}$, where Γ^j_{mk} is the Levi-Civita connection associated with g^{ij}: $\Gamma^k_{ij} = (1/2)\sum_{s=1}^n g^{ks}(\partial g^{sj}/\partial x^i + \partial g^{is}/\partial x^j - \partial g^{ij}/\partial x^s)$.

For P (11) and $\tau = \tau(\boldsymbol{u})$ we have (see e.g. [13])

$$(L_\tau(P))^{ij} = \sum_{s=1}^{n}\left(\tau^s\frac{\partial g^{ij}}{\partial u^s} - g^{sj}\frac{\partial\tau^i}{\partial u^s} - g^{is}\frac{\partial\tau^j}{\partial u^s}\right)D + \sum_{s,k=1}^{n}\left(\tau^s\frac{\partial b^{ij}_k}{\partial u^s} - b^{sj}_k\frac{\partial\tau^i}{\partial u^s} - b^{is}_k\frac{\partial\tau^j}{\partial u^s} + b^{ij}_s\frac{\partial\tau^s}{\partial u^k} - g^{is}\frac{\partial^2\tau^j}{\partial u^s\partial u^k}\right)u^k_x.$$

Proposition 5 *Let P be a Hamiltonian operator of the form (11) with $\det g^{ij} \neq 0$. Then a skew-symmetric operator $\tilde{P}$ of Dubrovin–Novikov type is a Hamiltonian operator compatible with P if and only if there exist local vector fields $\tau = \tau(\boldsymbol{u})$ and $\tilde{\tau} = \tilde{\tau}(\boldsymbol{u})$ on N such that locally we have $\tilde{P} = L_\tau(P)$ and $L^2_\tau(P) = L_{\tilde{\tau}}(P)$.*

Let us stress that τ and $\tilde{\tau}$ depend here solely on $\boldsymbol{u}$. In particular, they do not involve $\boldsymbol{u}_j$ with $j \geq 1$. Proposition 5 is proved along the very same lines as Proposition 4, with usage of the following lemma.

Lemma 1 *Suppose that P is a Hamiltonian operator of the form (11) with $\det g^{ij} \neq 0$. Then for any skew-symmetric operator $\tilde{P}$ of Dubrovin–Novikov type (not necessarily Hamiltonian) that satisfies $[\tilde{P}, P] = 0$ there exists a local vector field $\tau = \tau(\boldsymbol{u})$ on N such that $\tilde{P}$ can be written locally as $\tilde{P} = L_\tau(P)$.*

Proof of the lemma. According to [31, 32] (see also [41] for a more formal setting) any Hamiltonian operator P of the form (11) with $\det g^{ij} \neq 0$ can be locally transformed into a Hamiltonian operator with constant coefficients of the form

$$P_{\text{can}}^{ij} = g_0^{ij} D, \tag{12}$$

where $g_0^{ij} = 0$ for $i \neq j$ and g_0^{ii} satisfy $(g_0^{ii})^2 = 1$, via an invertible transformation $\boldsymbol{u} \mapsto \tilde{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{u})$. But for $P = P_{\text{can}}$ our lemma holds by Proposition 1 of [42], cf. also [13], and we obtain the desired result in full generality by just going back from $\tilde{\boldsymbol{u}}$ to $\boldsymbol{u}$. □

Thus, the classification of compatible Hamiltonian operators of Dubrovin–Novikov type essentially amounts, at least locally, to the classification of pairs of vector fields τ and $\tilde{\tau}$ on N such that $L_\tau^2(P) = L_{\tilde{\tau}}(P)$ for $P = P_{\text{can}}$ (12). In particular, in this way we can recover the quasihomogeneous Hamiltonian pairs constructed by Dubrovin in [43] using the theory of Frobenius manifolds. For $\tilde{\tau} = 0$ we come back to the case analysed by Fordy and Mokhov [12]. The comparison of the classification results obtained in our approach with e.g. those of Mokhov [44] will be the subject of our future work.

Consider now the systems of hydrodynamic type, that is, the systems of the form $\boldsymbol{u}_t = A(\boldsymbol{u})\boldsymbol{u}_x$, where $A(\boldsymbol{u})$ is an $n \times n$ matrix [31, 32]. Proposition 5 immediately yields the following analog of Corollary 5.

Corollary 6 *Consider a Poisson bivector P of Dubrovin–Novikov type with $\det g^{ij} \neq 0$ and an evolutionary vector field $X = (A(\boldsymbol{u})\boldsymbol{u}_x)\partial/\partial\boldsymbol{u}$ such that $L_X(P) = 0$. Then X is locally bi-Hamiltonian with respect to P and some other Poisson bivector $\tilde{P}$ of Dubrovin–Novikov type if and only if there exist local vector fields $\tau(\boldsymbol{u})$ and $\tilde{\tau}(\boldsymbol{u})$ on N such that locally we have $L_\tau(P) \neq 0$, $L_X(L_\tau(P)) = 0$, $L_\tau^2(P) = L_{\tilde{\tau}}(P)$, and $\tilde{P}$ can be locally written as $L_\tau(P)$.*

Remark 2 In analogy with Remark 1, let us call a Poisson pencil $\mathcal{P}$ associated with compatible Hamiltonian operators P_0, P_1 of Dubrovin–Novikov type *regular* if there exist constants λ and μ such that $P = \lambda P_0 + \mu P_1$ has $\det g^{ij} \neq 0$. Let $\tilde{P} = \tilde{\lambda} P_0 + \tilde{\mu} P_1$ be linearly independent of P. Then we can restate Corollary 6 as follows:

An evolutionary vector field $X = (A(\boldsymbol{u})\boldsymbol{u}_x)\partial/\partial\boldsymbol{u}$ is locally bi-Hamiltonian with respect to a regular Poisson pencil $\mathcal{P}$ of Hamiltonian operators of Dubrovin–Novikov type if and only if $L_X(P) = 0$ and there exist local vector fields τ and $\tilde{\tau}$ on N such that locally we have $L_\tau(P) \neq 0$, $L_X(L_\tau(P)) = 0$, $L_\tau^2(P) = L_{\tilde{\tau}}(P)$, and $\tilde{P}$ can be locally written as $L_\tau(P)$.

This reformulation of Corollary 6 gives necessary and sufficient conditions for existence of a regular Poisson pencil of Hamiltonian operators of Dubrovin–Novikov type such that a given EVF $(A(\boldsymbol{u})\boldsymbol{u}_x)\partial/\partial\boldsymbol{u}$ is locally bi-Hamiltonian with respect to this Poisson pencil (cf. Remark 1). Moreover, we expect that this result can be efficiently employed for the classification of such EVFs and associated bi-Hamiltonian hydrodynamic type systems, and we plan to address this issue elsewhere.

7 Conclusions

We have shown above that $[L_\tau^2(P), P] = 0$ is a necessary and sufficient condition for $\tilde{P} = L_\tau(P)$ to be a Poisson bivector, if so is P. This enabled us to benefit from the powerful results of [27] on the Lichnerowicz–Poisson cohomology and obtain new easily verifiable necessary and sufficient conditions for existence of Poisson bivectors compatible with a given Poisson bivector and for transformability of a given dynamical system into a bi-Hamiltonian form.

For instance, when P and $\tilde{P}$ have a common scaling, the verification of their compatibility and of the conditions for $\tilde{P} = L_\tau(P)$ to be a Poisson bivector along the lines of Proposition 2 and Corollary 3 is considerably easier than the cumbersome direct computation of the corresponding Schouten brackets, especially in the infinite-dimensional case, as the application of our results requires just the computation of Lie derivatives that can be readily performed using the computer algebra software like *Maple* or *Mathematica*.

Moreover, unlike e.g. Theorem 5.1 of [24], our results provide not merely sufficient but *necessary and sufficient* conditions ensuring that for a given dynamical system $\dot{x} = X$ there exists a regular or weakly regular Poisson pencil such that $\dot{x} = X$ is locally bi-Hamiltonian with respect to this pencil, see Remark 1 for details. We believe that this result will enable one to perform a complete classification of such dynamical systems.

The results of Propositions 3, 4 and 5 enable us to perform, at least in certain cases, a 'partial integration' of the system $[\tilde{P}, P] = [\tilde{P}, \tilde{P}] = 0$ by replacing it with $\tilde{P} = L_\tau(P)$ and $L^2_\tau(P) = L_{\tilde{\tau}}(P)$. The latter system is much easier to solve just because it is generally easier to find a couple of vectors than a skew-symmetric rank two tensor, cf. [24].

Thus, we can classify the Poisson bivectors compatible with a given Poisson bivector P of locally constant rank with $\dim\ker P \leq 1$ using Propositions 3 and 4 after bringing P into canonical form using the Darboux theorem. We expect that a similar approach, based on the local description of $H^2_P(M)$, could be extended to the case of $\dim\ker P > 1$ as well. The classification of Hamiltonian operators of Dubrovin–Novikov type also can be performed in a similar fashion, see Section 6 above. It would be interesting to compare the results obtained in this way with those found by other known classification methods, for instance from [8]–[11], and [44], and to find out whether one can extend the results of present paper to the case of Dirac [5] and Jacobi (see e.g. [45]) structures. We plan to address these issues in our future work.

Acknowledgements

I am sincerely grateful to Profs. M. Błaszak, L. Hlavatý, P.J. Olver, V.V. Sokolov and Drs. E.V. Ferapontov, M.V. Pavlov, M. Pedroni and R.G. Smirnov for stimulating discussions. I am also pleased to thank Profs. M. Błaszak, C. Morosi, W. Oevel, F. Turiel and Drs. M. Pedroni, F. Petalidou and R.G. Smirnov for kindly providing me with the copies of [7], [3, 19], [17], [9], [22], [25], [18, 24], respectively. Last but not least, I express deep gratitude to the organizers of 11^{th} Colloquium "Quantum Groups and Integrable Systems" (June 20-22, 2002, Prague, Czech Republic) where most of the results of this paper were presented, for their kind hospitality.

This research was supported in part by DFG via Graduiertenkolleg "Geometrie und Nichtlineare Analysis" at Institut für Mathematik of Humboldt-Universität zu Berlin, where the author held a postdoctoral fellowship in 2001, and by the Jacob Blaustein postdoctoral fellowship. I also acknowledge with gratitude the support from the Czech Grant Agency under grant No. 201/00/0724 and the Ministry of Education, Youth and Sports of Czech Republic under grant MSM:J10/98:192400002.

References

[1] Magri, F.: A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (1978), 1156–1162.

[2] Gelfand, I.M., Dorfman, I.Ya.: Hamiltonian operators and algebraic structures related to them, *Funct. Anal. Appl.* **13** (1979), 248–262.

[3] Magri, F. and Morosi, C.: A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson–Nijenhuis manifolds, *Quaderno* **S 19** (1984), Dip. Matematica, Università degli Studi di Milano, Milan, 1984.

[4] Fuchssteiner, B., and Fokas, A.S.: Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D* **4** (1981/82), no. 1, 47–66.

[5] Dorfman, I.: *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley & Sons, Chichester, 1993.

[6] Olver, P.J.: *Applications of Lie Groups to Differential Equations*, Springer, New York, 1993.

[7] Błaszak, M.: *Multi-Hamiltonian Theory of Dynamical Systems*, Springer, Heidelberg, 1998.

[8] Gelfand, I.M. and Zakharevich, I.: On the local geometry of a bi-Hamiltonian structure, in: L. Corwin (ed), *The Gelfand Seminars, 1990-1992*, Birkhäuser, Basel, pp. 51–112.

[9] Turiel, F.: Classification locale simultanée de deux formes symplectiques compatibles, *Manuscripta Math.* **82** (1994), no. 3-4, 349–362.

[10] Turiel, F.: Tissus de Veronese analytiques de codimension supérieure et structures bihamiltoniennes, *C. R. Acad. Sci. Paris* Sér. I Math. **331** (1) (2000), no. 1, 61–64.

[11] Ferapontov, E.V.: Compatible Poisson brackets of hydrodynamic type, *J. Phys. A: Math. Gen.* **34** (2001), 2377–2388, arXiV: `math.DG/0005221`

[12] Fordy, A.P., Mokhov O.I.: On a special class of compatible Poisson structures of hydrodynamic type, *Physica D* **152/153** (2001), 475–490.

[13] Mokhov, O.I.: Compatible Dubrovin–Novikov Hamiltonian operators, Lie derivative and integrable systems of hydrodynamic type, to appear in *Nonlinear evolution equations and dynamical systems (Cambridge, July 24-30, 2001)* (*Preprint* `math.DG/0201281`)

[14] Kosmann-Schwarzbach, Y., Magri, F.: Lax-Nijenhuis operators for integrable systems. *J. Math. Phys.* **37** (1996), no. 12, 6173–6197.

[15] Olver, P.J.: Canonical forms and integrability of bi-Hamiltonian systems, *Phys. Lett. A* **148** (1990), no. 3-4, 177–187.

[16] Fernandes, R.L.: Completely integrable bi-Hamiltonian systems, *J. Dynam. Differential Equations* **6** (1994), no. 1, 53–69.

[17] Oevel, W.: *Rekursionmechanismen für Symmetrien und Erhaltungssätze in Integrablen Systemen*, Ph.D. thesis, University of Paderborn, Paderborn, 1984.

[18] Oevel, W.: A geometrical approach to integrable systems admitting time dependent invariants, in M. Ablowitz, B. Fuchssteiner, M. Kruskal (eds) *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations (Oberwolfach, 1986)* World Scientific Publishing, Singapore, 1987, pp. 108–124.

[19] Morosi, C.: Some remarks on the construction of bi-Hamiltonian manifolds. *Atti Sem. Mat. Fis. Univ. Modena* **36** (1988), no. 2, 303–320.

[20] Magnano, G., Magri, F.: Poisson–Nijenhuis structures and Sato hierarchy, *Rev. Math. Phys.* **3** (1991), no. 4, 403–466.

[21] Magri, F.: On the geometry of soliton equations, *Acta Appl. Math.* **41** (1995), 247–270.

[22] Casati, P., Falqui, G., Magri, F., Pedroni, M., The KP Theory revisited II. The reduction theory and the Adler–Gel'fand–Dickey brackets, Preprint SISSA/3/96/FM, Trieste 1996.

[23] Falqui, G., Magri, F., Pedroni, M., Bi-Hamiltonian geometry, Darboux coverings, and linearization of the KP hierarchy, *Comm. Math. Phys.* **197** (1998), 303–324, arXiv: `solv-int/9806002`

[24] Smirnov, R.G.: Bi-Hamiltonian formalism: a constructive approach, *Lett. Math. Phys.* **41** (1997), 333–347.

[25] Petalidou, F.: *Étude locale de structures bihamiltoniennes*, Thèse de Doctorat, Université Pierre et Marie Curie, Paris, 1998.

[26] Petalidou, F.: Sur la symplectisation de structures bihamiltoniennes, *Bull. Sci. Math.* **124**(2000), no. 4, 255–286.

[27] Lichnerowicz, A.: Les variétés de Poisson et leurs algèbres de Lie associées, *J. Differential Geometry* **12** (1977), no. 2, 253–300.

[28] Fokas, A.S., Fuchssteiner, B.: The hierarchy of the Benjamin-Ono equation. *Phys. Lett. A* **86**(1981), no. 6-7, 341–345.

[29] Sergyeyev, A.: A simple way to make a Hamiltonian system into bi-Hamiltonian one, Preprint GA 4/2002, Silesian University in Opava, Czech Republic, May 2002.

[30] Chavchanidze, G.: Non-Noether symmetries and their influence on phase space geometry, *J. Geom. Phys.* **48** (2003), 190–202, arXiV: `math-ph/0211014`

[31] Dubrovin, B.A., Novikov, S.P.: Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method, *Soviet Math. Dokl.* **27** (1983), 665–669.

[32] Dubrovin, B.A., Novikov, S.P.: On Poisson brackets of hydrodynamic type, *Soviet Math. Dokl.* **30** (1984), 651–654.

[33] Błaszak, M.: Degenerate Poisson pencils on curves: new separability theory, *J. Nonlinear Math. Phys.* **7** (2000), no. 2, 213–243, arXiv: `math.DG/9803106`

[34] Błaszak, M.: From bi-Hamiltonian geometry to separation of variables: stationary Harry-Dym and the KdV dressing chain, *J. Nonlinear Math. Phys.* **9** (2002), Supplement, no. 2, 1–13, arXiv: `nlin.SI/0202001`

[35] Błaszak, M., and Wojciechowski, S.: Bi-Hamiltonian dynamical systems related to low-dimensional Lie algebras, *Physica A* **155** (1989), no. 3, 545–564.

[36] Vaisman, I.: *Lectures on the geometry of Poisson manifolds*, Birkhäuser, Basel, 1994.

[37] Tondo, G. and Morosi, C.: Bi-Hamiltonian manifolds, quasi-bi-Hamiltonian systems and separation variables, *Rep. Math. Phys.* **44** (1999), no. 1-2, 255–266, arXiv: `solv-int/9811008`

[38] Krasil'shchik, I. S.: Cohomological approach to Poisson structures on nonlinear evolution equations, *Lobachevskii J. Math.* **3** (1999), 127–145.

[39] Ferapontov, E.V.: Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type, *Funct. Anal. Appl.* **25** (1991), 195–204.

[40] Kupershmidt, B.A.: Mathematics of dispersive water waves, *Comm. Math. Phys.* **99** (1985), no. 1, 51–73.

[41] Getzler, E.: A Darboux theorem for Hamiltonian operators in the formal calculus of variations. *Duke Math. J.* **111** (2002), no. 3, 535–560, arXiv: `math.DG/0002164`

[42] Degiovanni, L., Magri, F., Sciacca, V.: *On deformation of Poisson manifolds of hydrodynamic type*, Preprint `nlin.SI/0103052` (2001)

[43] Dubrovin, B.A.: Flat pencils of metrics and Frobenius manifolds, in: *Integrable systems and algebraic geometry*, Saito M.-H. (ed.) et al., World Scientific, Singapore, 1998, pp. 47–72, arXiv: `math.DG/9803106`

[44] Mokhov, O.I.: On the integrability of equations for nonsingular pairs of compatible flat metrics, *Theoret. and Math. Phys.* **130** (2002), no. 2, 198–212.

[45] Guedira, F., Lichnerowicz, A.: Géométrie des algèbres de Lie locales de Kirillov, *J. Math. Pures Appl.* **63** (1984), no. 4, 407–484.